The action of $\mathbf{SL}_2$ on abelian varieties

Arnaud BEAUVILLE

Introduction

The title is somewhat paradoxical: we know that a linear group can only act trivially on an abelian variety. However we also know that there are not enough morphisms in algebraic geometry, a problem which may be fixed sometimes by considering *correspondences* between two varieties – that is, algebraic cycles on their product, modulo rational equivalence. Our main result is the construction of a natural morphism of the algebraic group $\mathbf{SL}_2$ into the group $\mathrm{Corr}(A)^*$ of (invertible) self-correspondences of any polarized abelian variety A. As a consequence the group $\mathbf{SL}_2$ acts on the $\mathbf{Q}$-vector space $\mathrm{CH}(A)$ parametrizing algebraic cycles (with rational coefficients) modulo rational equivalence, in such a way that this space decomposes as the direct sum of irreducible finite-dimensional representations. This gives various results of Lefschetz type for the Chow group.

This action of $\mathbf{SL}_2$ on $\mathrm{CH}(A)$ is already known: it appears implicitly in the work of Künnemann [K], and explicitely in the unpublished thesis [P2]. But though it has been repeatedly used in recent work on the subject ([P3], [P4], [Mo]), a detailed exposition does not seem to be available in the literature. The aim of this paper is to fill this gap, and also to explain the link with the action of $\mathrm{SL}_2(\mathbf{Z})$ on the derived category $\mathbf{D}(A)$ found by Mukai [M].

In section 1 we recall some classical facts on correspondences, mainly to fix our notations and conventions. In section 2 we explain how to deduce from Mukai's results a homomorphism of $\mathrm{SL}_2(\mathbf{Z})$ into $\mathrm{Corr}(A)^*$, hence an action of $\mathrm{SL}_2(\mathbf{Z})$ onto $\mathrm{CH}(A)$. In sections 3 and 4 we show that these extend to $\mathbf{SL}_2$, using a description of this algebraic group by generators and relations due to Demazure. In section 5 we deduce some applications; the most interesting perhaps is a twisted version of the hard Lefschetz theorem for $\mathrm{CH}(A)$: if $\theta \in \mathrm{CH}^1(A)$ is an ample symmetric class, the multiplication map $\times\, \theta^{g-2p+s} : \mathrm{CH}_s^p(A) \to \mathrm{CH}_s^{g-p+s}(A)$ is an isomorphism (the subscript s refers to the decomposition of $\mathrm{CH}(A)$ into eigensubspaces for the operators n_A^*, $n \in \mathbf{Z}$, see (1.5) below).

1. Correspondences

(1.1) We fix an algebraically closed field k. We denote by A a smooth projective variety over k; from (1.4) on A will be an abelian variety.

As mentioned in the introduction, we will denote by $CH(A)$ the Chow group of algebraic cycles *with rational coefficients* on A modulo rational equivalence[1]. We briefly recall some basic facts about correspondences, referring to [F] for a detailed treatment.

A *correspondence* of A is an element of $CH(A \times A)$. If α, β are two correspondences, we define their composition by $\beta \circ \alpha = (p_{13})_*(p_{12}^*\alpha \cdot p_{23}^*\beta)$, where $p_{ij} : A \times A \times A \to A \times A$ is the projection on the i-th and j-th factors. This defines an (associative) **Q**-algebra structure on $CH(A \times A)$; this algebra is denoted $Corr(A)$, and its subgroup of invertible elements by $Corr(A)^*$. The unit element is the class $[\Delta_A]$ of the diagonal in $A \times A$.

To a class α in $Corr(A)$ we associate a **Q**-linear map

$$\alpha_* : CH(A) \to CH(A) \quad \text{defined by} \quad \alpha_* z = q_*(\alpha \cdot p^* z) \ ,$$

where p and q are the two projections from $A \times A$ to A. This gives a **Q**-algebra homomorphism $Corr(A) \to End_{\mathbf{Q}}(CH(A))$, hence a group homomorphism $Corr(A)^* \to Aut_{\mathbf{Q}}(CH(A))$.

(1.2) We will need a few formulas satisfied by correspondences; they can be found in (or follow easily from) [F], § 16.

(1.2.*a*) Let $\Delta : A \hookrightarrow A \times A$ be the diagonal morphism. We have

$$\Delta_* z \circ \alpha = \alpha \cdot q^* z \qquad \alpha \circ \Delta_* z = \alpha \cdot p^* z \quad \text{for } \alpha \in Corr(A), \ z \in CH(A) \ ;$$

taking $\alpha = \Delta_* x$ we see that *the map* $\Delta_* : CH(A) \to Corr(A)$ *is a* **Q**-*algebra homomorphism.*

(1.2.*b*) Let u be an endomorphism of A, Γ_u the class of its graph, and Γ'_u its transpose $(= (u, 1_A)_* \Delta_A)$. For $\alpha \in Corr(A)$, we have

$$\Gamma_u \circ \alpha = (1, u)_* \alpha \ , \quad \Gamma'_u \circ \alpha = (1, u)^* \alpha \ , \quad \alpha \circ \Gamma_u = (u, 1)^* \alpha \ , \quad \alpha \circ \Gamma'_u = (u, 1)_* \alpha \ .$$

(1.3) The above constructions work almost word for word replacing $CH(A)$ by the derived category $\mathbf{D}(A)$ of bounded complexes of coherent sheaves on A. A

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[1] We could replace rational equivalence by any adequate equivalence relation, see [S].

derived correspondence of A is an object of $\mathbf{D}(A \times A)$. Using the notations of (1.1), we define the composition of two such objects K, L as $L \circ K := (p_{13})_*(p_{12}^* K \otimes p_{23}^* L)$. This defines an associative multiplication on the set of isomorphism classes of objects of $\mathbf{D}(A \times A)$; we will denote this monoid by $\mathrm{Dcorr}(A)$, and by $\mathrm{Dcorr}(A)^*$ the subgroup of invertible elements. Their unit is the sheaf $\mathcal{O}_{\Delta_A}$.

As above we associate to $K \in \mathrm{Dcorr}(A)$ the Fourier-Mukai transform $\Phi_K :$ $\mathbf{D}(A) \to \mathbf{D}(A)$, defined by $\Phi_K(\star) = q_*(p^*(\star) \otimes K)$. This defines a group homomorphism

$$\Phi : \mathrm{Dcorr}(A)^* \to \mathrm{Aut}(\mathbf{D}(A))$$

where $\mathrm{Aut}(\mathbf{D}(A))$ is the group of isomorphism classes of auto-equivalences of the triangulated category $\mathbf{D}(A)$. By a celebrated theorem of Orlov [O], this map is *bijective.*

(1.4) From now on we assume that A is an abelian variety. In that case the constructions of (1.1) and (1.3) are linked by the Chern character, which is a monoid homomorphism $\mathrm{ch} : \mathrm{Dcorr}(A) \to \mathrm{Corr}(A)$. We have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Dcorr}(A)^* & \xrightarrow{\;\Phi\;} & \mathrm{Aut}(\mathbf{D}(A)) \\
{\scriptstyle \mathrm{ch}} \downarrow & & \downarrow {\scriptstyle \kappa} \\
\mathrm{Corr}(A)^* & \xrightarrow{\;*\;} & \mathrm{Aut}_{\mathbf{Q}}(\mathrm{CH}(A))
\end{array}
$$

where the map κ is defined as follows: an automorphism of $\mathbf{D}(A)$ induces an automorphism of the K-theory group $K(A)$, hence a $\mathbf{Q}$-linear automorphism of $K(A) \otimes \mathbf{Q}$, hence a $\mathbf{Q}$-linear automorphism of $\mathrm{CH}(A)$ via the isomorphism $\mathrm{ch} : K(A) \otimes \mathbf{Q} \xrightarrow{\sim} \mathrm{CH}(A)$.

(1.5) For an abelian variety A, the unit element $[\Delta_A]$ of $\mathrm{Corr}(A)$ has a canonical decomposition as a sum of orthogonal idempotents ([D-M], Theorem 3.1)

$$[\Delta_A] = \sum_{i=0}^{2g} \pi_i$$

characterized by the property $(1, k)^* \pi_i = k^i \pi_i$. This decomposition induces a grading $\mathrm{CH}(A) = \underset{s}{\oplus}\, \mathrm{CH}_s(A)$, with

$$\mathrm{CH}_s^p(A) = (\pi_{2p-s})_*(\mathrm{CH}^p(A)) = \{x \in \mathrm{CH}^p(A) \mid n_A^* x = n^{2p-s} x \ \text{ for all } n \in \mathbf{Z}\}$$

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(see [B2]).

(1.6) Suppose now that A has a polarization θ, which we will view as a *symmetric* (ample) element of $\mathrm{CH}^1(A)$. The polarization defines an isogeny $\varphi : A \to \hat{A}$. The *Poincaré bundle* $\mathcal{P}$ on $A \times A$ is by definition the pull back by $(1, \varphi)$ of the Poincaré bundle on $A \times \hat{A}$; it will play an important role in what follows.

Now there is a sign ambiguity in the definition of φ, hence of $\mathcal{P}$. Most authors, following Mumford in [Md], use the formula $\varphi(a) = \mathcal{O}_A(\Theta - \Theta_a)$, where Θ is a divisor defining the polarization and $\Theta_a = \Theta + a$ denotes its translate by $a \in A$. This convention has some serious drawbacks. One of them is that if A is the Jacobian of a curve C, the natural map $\hat{A} = \mathbf{Pic}^0(A) \to A = \mathbf{Pic}^0(C)$ deduced from the embedding of C in A (defined up to translation) is the *opposite* of φ^{-1}. More important for us, it leads to sign problems in the definition of the action of (a covering of) $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{D}(A)$. Because of these difficulties we will use the isomorphism $\varphi : A \to \hat{A}$ defined by $\varphi(a) = \mathcal{O}_A(\Theta_a - \Theta)$. With this convention, by the see-saw theorem the class in $\mathrm{CH}^1(A \times A)$ of the Poincaré bundle is $p^*\theta + q^*\theta - m^*\theta$, where $m : A \times A \to A$ is the addition map $(a, b) \mapsto a + b$.

2. The homomorphism $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{Corr}(A)^*$

(2.1) In [M] Mukai observes that the derived category $\mathbf{D}(A)$ of a principally polarized abelian variety (A, θ) carries an action of $\mathrm{SL}_2(\mathbf{Z})$ "up to shift". This is nicely elaborated in [P1] as an action of a central extension of $\mathrm{SL}_2(\mathbf{Z})$ by $\mathbf{Z}$, the trefoil group $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ (also known as the braid group on three strands). We will need only to describe this action in a naïve sense, that is, as a group homomorphism of $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ in the group $\mathrm{Aut}(\mathbf{D}(A))$ (1.3).

Recall that the group $\mathrm{SL}_2(\mathbf{Z})$ is generated by the elements

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with the relations $w^2 = (uw)^3$, $w^4 = 1$. The group $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ is generated by two elements $\tilde{u}, \tilde{w}$ with the relation $\tilde{w}^2 = (\tilde{u}\tilde{w})^3$; the covering $\widetilde{\mathrm{SL}}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z})$ is obtained by mapping $\tilde{u}$ to u and $\tilde{w}$ to w.

(2.2) Let $\mathcal{P}$ be the *Poincaré line bundle* on $A \times A$ (1.6). The functor $\Phi_{\mathcal{P}}$ is an autoequivalence of the category $\mathbf{D}(A)$ – this is the original Fourier-Mukai functor [M]. We choose a *symmetric* line bundle L on A with class θ and define an action

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of $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ on $\mathbf{D}(A)$ by mapping $\tilde{u}$ to the functor $\otimes L$ and $\tilde{w}$ to $\Phi_{\mathcal{P}}$. Theorem 3.13 of [M] gives[2] $(\Phi_{\mathcal{P}})^2 = (\otimes L \circ \Phi_{\mathcal{P}})^3 = (-1_A)^*[-g]$, so that we have indeed an action of $\widetilde{\mathrm{SL}}_2(\mathbf{Z})$ on $\mathbf{D}(A)$, with the central element $z = \tilde{w}^2 = (\tilde{u}\tilde{w})^3$ acting as $(-1_A)^*[-g]$; thus z^2 acts as the shift $[-2g]$.

(2.3) Observe that the functor $\otimes L$ can be written $\Phi_{\Delta_* L}$. By (1.3) it follows that the homomorphism $\widetilde{\mathrm{SL}}_2(\mathbf{Z}) \to \mathrm{Aut}(\mathbf{D}(A))$ factors through a group homomorphism

$$\widetilde{\mathrm{SL}}_2(\mathbf{Z}) \to \mathrm{Dcorr}(A)^*$$

mapping $\tilde{u}$ to $\Delta_* L$ and $\tilde{w}$ to $\mathcal{P}$.

(2.4) We now consider the composition $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{Dcorr}(A)^* \xrightarrow{\mathrm{ch}} \mathrm{Corr}(A)^*$, where ch is the Chern character (1.4). Since $\mathrm{ch}\, E[p] = (-1)^p \,\mathrm{ch}\, E$, this homomorphism maps z^2 to the unit element, hence factors as a homomorphism

$$\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{Corr}(A)^*$$

which maps u to $\Delta_* e^\theta$, and w to $e^{\wp}$, where $\wp$ is the class of $\mathcal{P}$ in $\mathrm{CH}^1(A)$.

(2.5) The argument extends with little change to the case of an arbitrary polarization. Let A be a polarized abelian variety, of dimension g; we denote by θ the unique symmetric element in $\mathrm{CH}^1(A)$ representing the polarization, and by $\wp = p^*\theta + q^*\theta - m^*\theta$ the class in $\mathrm{CH}^1(A \times A)$ of the Poincaré bundle (1.6). The *degree* of θ is $d = \frac{\theta^g}{g!}$.

Proposition 2.6.– *There is a* (*unique*) *group homomorphism* $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{Corr}(A)^*$ *mapping* u *to* $\Delta_* e^\theta$ *and* w *to* $d^{-1} e^{\wp}$.

Proof : We choose an isogeny π of A onto an abelian variety A_0 with a principal polarization θ_0 such that $\theta = \pi^*\theta_0$ ([Md], § 23, Cor. 1 of Thm. 4). One checks readily that the $\mathbf{Q}$-linear isomorphism $d^{-1}(\pi,\pi)^* : \mathrm{CH}(A_0 \times A_0) \to \mathrm{CH}(A \times A)$ is compatible with the composition of correspondences, thus induces an isomorphism of algebras $\mathrm{Corr}(A_0) \xrightarrow{\sim} \mathrm{Corr}(A)$. Let Δ_0 denote the diagonal morphism of A_0. We have $(\pi,\pi) \circ \Delta = \Delta_0 \circ \pi$ and $f \circ (\pi,\pi) = \pi \circ f$ if $f = p, q$ or m. From this one easily gets

$$(\pi,\pi)^* \Delta_* e^{\theta_0} = d \Delta_* e^\theta \quad \text{and} \quad (\pi,\pi)^* e^{p^*\theta_0 + q^*\theta_0 - m^*\theta_0} = e^{p^*\theta + q^*\theta - m^*\theta} \,,$$

hence the result. ∎

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[2]   Note that our functor $\Phi_{\mathcal{P}}$ is Mukai's $R\mathcal{S}$ composed with $(-1_A)^*$.

3. Extension to $\mathbf{SL}_2$

We will now show that the homomorphism $SL_2(\mathbf{Z}) \to \mathrm{Corr}(A)^*$ extends to the algebraic group $\mathbf{SL}_2$ over $\mathbf{Q}$. The essential tool is the description of $\mathbf{SL}_2$ by generators and relations given (in a much more general set-up) in [D], Theorem 6.2.

We denote by B the upper triangular Borel subgroup of $\mathbf{SL}_2$. We still denote by w and u the elements $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ of $SL_2(\mathbf{Q})$. By a $\mathbf{Q}$-group we mean a sheaf of groups over $\mathrm{Spec}(\mathbf{Q})$ for the fppf topology.

Proposition 3.1.– *Let* H *be a* $\mathbf{Q}$*-group. Suppose given a morphism of* $\mathbf{Q}$*-groups* $\beta : \mathrm{B} \to \mathrm{H}$ *and an element* $h \in \mathrm{H}(\mathbf{Q})$. *Assume that:*
   (i) $h\beta(t)h^{-1} = \beta(t^{-1})$ *for* t *in the maximal torus of* B *;*
   (ii) $h^2 = (\beta(u)h)^3 = \beta(-\mathrm{I})$ *in* $\mathrm{H}(\mathbf{Q})$.
Then there is a (unique) morphism of $\mathbf{Q}$*-groups* $\varphi : \mathbf{SL}_2 \to \mathrm{H}$ *extending* β *and mapping* w *to* h.

Proof: This is [D], Theorem 6.2, in the case $\mathrm{S} = \mathrm{Spec}(\mathbf{Q})$, $\mathrm{G} = \mathbf{SL}_2$ (note that our element w is the opposite of the one in *loc. cit.*). ∎

Observe that (ii) can be rephrased by saying that there is a homomorphism $SL_2(\mathbf{Z}) \to \mathrm{H}(\mathbf{Q})$ which maps w to h and coincides with β on $\mathrm{B}(\mathbf{Z})$.

(3.2) If C is a $\mathbf{Q}$-algebra, the functor $\mathrm{R} \mapsto (\mathrm{C} \otimes_{\mathbf{Q}} \mathrm{R})^*$ is a $\mathbf{Q}$-group; its Lie algebra is C, endowed with the bracket $[x, y] = xy - yx$. We will denote by $\mathbf{Corr}^*(A)$ the $\mathbf{Q}$-group obtained from the $\mathbf{Q}$-algebra $\mathrm{Corr}(A)$ in this way.

Let $\delta : \mathrm{A} \times \mathrm{A} \to \mathrm{A}$ be the difference map $(a, b) \mapsto b - a$. For $n \in \mathbf{Z}$ we denote by $\Gamma_n \in \mathrm{Corr}(A)$ the graph of the multiplication by n, and by Γ'_n its transpose.

Theorem 3.3.– *Let* A *be an abelian variety, of dimension* g, *with a polarization* θ *of degree* d. *There is a morphism of* $\mathbf{Q}$*-groups* $\varphi : \mathbf{SL}_2 \to \mathbf{Corr}^*(A)$ *such that, for* $n \in \mathbf{Z} - \{0\}$, $a \in \mathbf{Q}$:

$$\varphi\left(\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix}\right) = n^{-g}\Gamma'_n \quad , \quad \varphi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) = d^{-1}e^{\wp} \, ,$$

$$\varphi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) = \Delta_* e^{a\theta} \quad , \quad \varphi\left(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\right) = d^{-1}a^g \, e^{\delta^* \theta / a} \, .$$

The corresponding Lie algebra homomorphism $\mathrm{L}\varphi : \mathfrak{sl}_2(\mathbf{Q}) \to \mathrm{Corr}(A)$ *is given, in the standard basis* $(\mathrm{X}, \mathrm{Y}, \mathrm{H})$ *of* $\mathfrak{sl}_2(\mathbf{Q})$, *by:*

$$\mathrm{L}\varphi(\mathrm{X}) = \Delta_*\theta \quad , \quad \mathrm{L}\varphi(\mathrm{Y}) = \frac{\delta^*\theta^{g-1}}{d(g-1)!} \quad , \quad \mathrm{L}\varphi(\mathrm{H}) = \sum_i (i - g)\pi_i \, .$$

Proof : We apply Proposition 3.1 with $H = \mathbf{Corr}^*(A)$. To define β we write B as a semi-direct product $\mathbf{G}_a \rtimes \mathbf{G}_m$. We define $\alpha : \mathbf{G}_a \to \mathbf{Corr}^*(A)$ by $a \mapsto \Delta_* e^{a\theta}$; this is a morphism of $\mathbf{Q}$-groups by $(1.2.a)$.

We define $\tau : \mathbf{G}_m \to \mathbf{Corr}^*(A)$ by $\tau(t) = t^{-g} \sum_i t^i \pi_i$, where (π_i) is the family of orthogonal idempotents considered in (1.5). This is a morphism of $\mathbf{Q}$-groups; for $t \in \mathbf{Z}$ we have $\tau(t) = t^{-g}(1,t)^* \Delta_A = t^{-g} \Gamma'_t$ and $\tau(t^{-1}) = t^g (\Gamma'_t)^{-1} = t^{-g} \Gamma_t$.

To ensure that $\beta = (\alpha, \tau) : \mathbf{G}_a \rtimes \mathbf{G}_m \to \mathbf{Corr}^*(A)$ is a morphism of groups, we must check the commutation relation $\tau(t)\, \alpha(a)\, \tau(t)^{-1} = \alpha(t^2 a)$. This relation is polynomial in t, so it suffices to check it for $t = s^{-1}$ with $s \in \mathbf{Z}$. In that case we have by $(1.2.b)$:

$$\tau(t)\, \alpha(a)\, \tau(t)^{-1} = s^{-2g} \Gamma_s \circ \Delta_* e^{a\theta} \circ \Gamma'_s = s^{-2g}(s,s)_* \Delta_* e^{a\theta} = s^{-2g} \Delta_* s_* e^{a\theta} \ ;$$

since $s_* \theta^p = s^{2g-2p} \theta^p$, this gives $\tau(t)\, \alpha(a)\, \tau(t)^{-1} = \Delta_* e^{t^2 a\theta} = \alpha(t^2 a)$ as required.

We take for h the element $d^{-1} e^{\wp}$ of $\mathrm{Corr}(A)^*$. Condition (ii) is then satisfied because h and $\beta(u)$ are the images of w and u by the homomorphism $\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{Corr}(A)^*$ (Prop. 2.6).

Condition (i) can be written $\tau(t) h \tau(t) = h$. Again it suffices to check the equality $\Gamma'_t \circ e^{\wp} \circ \Gamma'_t = t^{2g} e^{\wp}$ for $t \in \mathbf{Z}$. The Poincaré bundle $\mathcal{P}$ satisfies $(t,1)^* \mathcal{P} = (1,t)^* \mathcal{P} = \mathcal{P}^t$ for every $t \in \mathbf{Z}$. Therefore $(t,1)^* \wp = (1,t)^* \wp$, and

$$\Gamma'_t \circ e^{\wp} \circ \Gamma'_t = (t,1)_* (t,1)^* e^{\wp} = t^{2g} e^{\wp} \ .$$

This proves the existence of φ satisfying the three first formulas stated. To prove the fourth one, put $v = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$; we have $v = uwu$. Thus, using $(1.2.a)$,

$$\varphi(v) = \Delta_* e^{\theta} \circ e^{\wp} \circ \Delta_* e^{\theta} = e^{\wp + p^* \theta + q^* \theta} \ .$$

Since $\wp = p^* \theta + q^* \theta - m^* \theta$ and $m^* \theta + \delta^* \theta = 2p^* \theta + 2q^* \theta$ by the seesaw theorem, this gives $\varphi(v) = e^{\delta^* \theta}$.

Now we use the equality $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t^{-2} & 1 \end{pmatrix}$ to get, for $t \in \mathbf{Z}$,

$$\varphi\left(\begin{pmatrix} 1 & 0 \\ t^{-2} & 1 \end{pmatrix} \right) = t^{-2g}\, \Gamma'_t \circ e^{\delta^* \theta} \circ \Gamma_t$$

We have by $(1.2.b)$

$$\Gamma'_t \circ e^{\delta^* \theta} \circ \Gamma_t = (t,t)^* e^{\delta^* \theta} = e^{\delta^* t^* \theta} = e^{t^2 \delta^* \theta} \ ;$$

putting $a = t^{-2}$ gives $\varphi\left(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \right) = d^{-1} a^g\, e^{\delta^* \theta / a}$ for $a = t^{-2}$ with $t \in \mathbf{Z}$, hence as usual for all a.

The value of $L\varphi$ follows from these formulas by differentiation. ∎

4. The action of $\mathbf{SL}_2$ on $\mathbf{CH}(A)$

(4.1) Let V be a $\mathbf{Q}$-vector space; the functor $R \mapsto GL(V \otimes R)$ on the category of (commutative) $\mathbf{Q}$-algebras is a $\mathbf{Q}$-group, that we will denote by $\mathbf{GL}(V)$. If G is an algebraic group over $\mathbf{Q}$, we define a representation of G on V as a morphism of $\mathbf{Q}$-groups $G \to \mathbf{GL}(V)$.

Recall that the *Pontryagin product* of two elements α, β of $CH(A)$ is defined by $\alpha * \beta := m_*(p^*\alpha \cdot q^*\beta)$.

We will denote by $\mathcal{F}$ the $\mathbf{Q}$-linear automorphism $d^{-1}(e^\wp)_*$ of $CH(A)$; this is the *Fourier transform* for Chow groups, see [B1] and [B2].

Theorem 4.2 .— *Let A be an abelian variety, with a polarization θ of degree d. There is a representation of $\mathbf{SL}_2$ on $CH(A)$, which is a direct sum of finite-dimensional representations, such that, for $n \in \mathbf{Z} - \{0\}$, $a \in \mathbf{Q}$, $z \in CH(A)$:*

$$\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix} \cdot z = n^{-g}n^*z \quad , \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot z = \mathcal{F}(z) \ ,$$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot z = e^{a\theta}z \quad , \quad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot z = d^{-1}a^g e^{\theta/a} * z \ .$$

The corresponding action of the Lie algebra $\mathfrak{sl}_2(\mathbf{Q})$ is given by:

$$\mathrm{X}z = \theta z \quad , \quad \mathrm{Y}z = d^{-1}\frac{\theta^{g-1}}{(g-1)!} * z \quad , \quad \mathrm{H}z = (2p - g - s)z \quad \text{for} \quad z \in CH_s^p(A) \ .$$

Proof : The homomorphism $\mathrm{Corr}(A) \to \mathrm{End}_\mathbf{Q}(CH(A))$ (1.1) defines a morphism of $\mathbf{Q}$-groups $\mathbf{Corr}^*(A) \to \mathbf{GL}(CH(A))$, hence by composition with φ a representation of $\mathbf{SL}_2$ on $CH(A)$. By definition, $g \cdot z = \varphi(g)_*z$ for $g \in \mathbf{SL}_2$, $z \in CH(A)$. Thus

$$\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix} \cdot z = n^{-g}(\Gamma_n)_*z = n^{-g}n^*z \, , \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot z = d^{-1}(e^\wp)_*z \overset{\text{def}}{=} \mathcal{F}(z) \, ,$$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot z = (\Delta_*e^{a\theta}z)_* = e^{a\theta}z \, , \quad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot z = d^{-1}a^g (e^{\delta^*\theta/a})_*z \, .$$

Let σ be the automorphism of $A \times A$ defined by $\sigma(a, b) = (b, a + b)$. We have $q \circ \sigma = m$, $p \circ \sigma = q$, $\delta \circ \sigma = p$, hence

$$(e^{\delta^*\theta/a})_*z = q_*\sigma_*\sigma^*(e^{\delta^*\theta/a} \cdot p^*z) = m_*(p^*e^{\theta/a} \cdot q^*z) = e^{\theta/a} * z \ .$$

In particular, H is diagonalizable and X, Y are nilpotent; this is enough to imply that V is a direct sum of finite-dimensional representations ([Bo], ch. 8, § 1, exerc. 4). ∎

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5. Application to Lefschetz type results

(5.1) In this section we will apply the well-known structure of finite-dimensional representations of $\mathbf{SL}_2$. We say that an element $z \in \mathrm{CH}_s^p(A)$ is *primitive* if $\theta^{g-1} * z = 0$. The primitive elements are exactly the lowest weight elements for the action of $\mathbf{SL}_2$ on $\mathrm{CH}(A)$.

Let $z \in \mathrm{CH}_s^p(A)$ be a primitive element. The subspace of $\mathrm{CH}(A)$ spanned (over $\mathbf{Q}$) by the $\theta^q z$ is an irreducible representation of $\mathbf{SL}_2$; it is identified with the space of polynomials in one variable of degree $\leq g + s - 2p$ (with the standard action) by the map $\mathrm{P} \mapsto \mathrm{P}(\theta)z$. This gives an explicit description of the action of $\mathbf{SL}_2$; in particular:

Proposition 5.2.— *If $z \in \mathrm{CH}_s^p(A)$ is primitive, we have $g + s - 2p \geq 0$, and $(z, \theta z, \ldots, \theta^{g+s-2p}z)$ is a basis of an irreducible subrepresentation of $\mathrm{CH}(A)$. The vector space $\mathrm{CH}(A)$ is a direct sum of subrepresentations of this type.* ∎

Corollary 5.3.— *Let $\mathrm{P}_s^p \subset \mathrm{CH}_s^p(A)$ be the subspace of primitive elements. Then $\mathrm{CH}_s^p(A) = \underset{q \leq p}{\oplus} \theta^{p-q} \mathrm{P}_s^q$.* ∎

Since the Fourier automorphism $\mathcal{F}$ of $\mathrm{CH}(A)$ is given by the action of w, we have:

Corollary 5.4.— *Let $z \in \mathrm{CH}_s^p(A)$ be a primitive element, and let $q \leq g + s - 2p$. Then $\mathcal{F}\left(\dfrac{\theta^q}{q!}z\right) = \dfrac{(-\theta)^r}{r!}z$, with $r = g + s - 2p - q$.* ∎

Proposition 5.5.— *The multiplication map $\times \theta^{q-p} : \mathrm{CH}_s^p(A) \to \mathrm{CH}_s^q(A)$ $(q \geq p)$ is injective for $p + q \leq g + s$ and surjective for $p + q \geq g + s$. In particular, it is bijective for $p + q = g + s$.*

Proof: Assume $p + q \leq g + s$; let $z \in \mathrm{CH}_s^p(A)$ with $\theta^{q-p}z = 0$. Using Cor. 5.3 we write $z = \sum_{r \leq p} \theta^{p-r} z_r$, with $z_r \in \mathrm{P}_s^r$; we have $\theta^{q-r}z_r = 0$ for each r. Since $q - r \leq g + s - 2r$ this implies $z_r = 0$ for each r, hence $z = 0$.

Assume $p + q \geq g + s$. To prove the surjectivity of $\times \theta^{q-p}$ it suffices, by Cor. 5.3, to prove that each nonzero element $\theta^{q-r} z_r$, with $z_r \in \mathrm{P}_s^r$, lies in the image. But since $\theta^{q-r} z_r \neq 0$ we have $q - r \leq g + s - 2r$, hence $q + r \leq g + s \leq p + q$ and finally $r \leq p$. Therefore $\theta^{q-r} z_r = \theta^{q-p}(\theta^{p-r} z_r)$. ∎

(5.6) In what follows we consider the filtration of $\mathrm{CH}(A)$ associated to the grading (1.5), that is, $\mathrm{F}^s \mathrm{CH}^p(A) := \underset{t \geq s}{\oplus} \mathrm{CH}_t^p(A)$. Conjecturally this is the Bloch-Beilinson filtration of $\mathrm{CH}(A)$, see [Mu].

Corollary 5.7.– *Let* $h \in \mathrm{CH}^1(\mathrm{A})$ *be an ample class. The multiplication map*

$$\times h^{q-p} : \mathrm{F}^{p+q-g}\mathrm{CH}^p(\mathrm{A}) \to \mathrm{F}^{p+q-g}\mathrm{CH}^q(\mathrm{A})$$

is injective.

Proof : Let θ be the component of h in $\mathrm{CH}^1_0(\mathrm{A})$; it is ample and symmetric. The map induced by $\times h^{q-p}$ on the associated graded spaces is

$$\times \theta^{q-p} : \underset{s \geq p+q-g}{\oplus} \mathrm{CH}^p_s(\mathrm{A}) \longrightarrow \underset{s \geq p+q-g}{\oplus} \mathrm{CH}^q_s(\mathrm{A}) \ ,$$

which is injective by Proposition 5.5; therefore $\times h^{q-p}$ is injective. ∎

Corollary 5.8.– *Let* D *be an ample divisor in* A. *The restriction map* $\mathrm{F}^{2p+1-g}\mathrm{CH}^p(\mathrm{A}) \to \mathrm{CH}^p(\mathrm{D})$ *is injective.*

Here $\mathrm{CH}^p(\mathrm{D})$ is the Chow group as defined in [F], chap. 2: $\mathrm{CH}^p(\mathrm{D}) = \mathrm{A}_{g-1-p}(\mathrm{D})$ in the notation of [F].

Proof : Let i be the natural injection of D in A, and let h be the class of D in $\mathrm{CH}^1(\mathrm{A})$. Let $z \in \mathrm{F}^{2p+1-g}\mathrm{CH}^p(\mathrm{A})$ such that $i^*z = 0$. Then $h \cdot z = i_*i^*z = 0$, so $z = 0$ by Cor. 5.7. ∎

(5.9) At this point we recall the *vanishing conjecture* of [B2]:

$$\mathrm{CH}_s(\mathrm{A}) = 0 \quad \text{for } s < 0 \ . \tag{5.9}$$

This implies $\mathrm{F}^0\mathrm{CH}(\mathrm{A}) = \mathrm{CH}(\mathrm{A})$, hence:

Proposition 5.10.– *Assume that* $\mathrm{CH}_s(\mathrm{A}) = 0$ *for* $s < 0$. *Then:*

a) *If* $h \in \mathrm{CH}^1(\mathrm{A})$ *is an ample class, the multiplication map* $\times h^{q-p} : \mathrm{CH}^p(\mathrm{A}) \to \mathrm{CH}^q(\mathrm{A})$ *is injective provided* $p + q \leq g$.

b) *If* D *is an ample divisor in* A, *the restriction map* $\mathrm{CH}^p(\mathrm{A}) \to \mathrm{CH}^p(\mathrm{D})$ *is injective for* $p \leq \frac{1}{2}(g-1)$. ∎

We have actually a more precise result, which has been shown to me by B. Fu (with a different proof, see [Fu]):

Proposition 5.11.– *Let* $h \in \mathrm{CH}^1(\mathrm{A})$ *be an ample class. The multiplication map* $\times h^{g-2p} : \mathrm{CH}^p(\mathrm{A}) \to \mathrm{CH}^{g-p}(\mathrm{A})$ *is injective for all* $p \leq \frac{g}{2}$ *if and only if conjecture* (5.9) *holds.*

Proof : It remains to prove that if $\mathrm{CH}_s(\mathrm{A})$ is nonzero for some $s < 0$, all multiplication maps cannot be injective. Let θ be the component of h in $\mathrm{CH}^1_0(\mathrm{A})$. By Cor. 5.3

there exists a nonzero primitive class $z \in \mathrm{CH}^p_s(\mathrm{A})$ for some integer $p \leq \frac{1}{2}(g+s)$; we have $g - 2p > g + s - 2p$ and therefore $\theta^{g-2p}z = 0$. The class h is equal to $\mathrm{T}^*_a\theta$ for some element $a \in \mathrm{A}$, where T_a denotes the translation $x \mapsto x + a$. We have $\mathrm{T}^*_a z \equiv z \mod \mathrm{F}^{s+1}\mathrm{CH}^p(\mathrm{A})$, hence $\mathrm{T}^*_a z \neq 0$, and $h^{g-2p}(\mathrm{T}^*_a z) = 0$, which proves our assertion. ∎

An interesting feature of Proposition 5.10 is that it makes sense for any smooth projective variety A; the same is true of Cor. 5.7, taking for $(\mathrm{F}^s\mathrm{CH}(\mathrm{A}))_{s \geq 0}$ the (conjectural) Bloch-Beilinson filtration of $\mathrm{CH}(\mathrm{A})$. These conjectures are thoroughly discussed in [Fu], where it is shown that they would follow from a weak version of the Beilinson conjectures.

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Arnaud BEAUVILLE

Institut Universitaire de France

&

Laboratoire J.-A. Dieudonné

UMR 6621 du CNRS

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Parc Valrose

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